Investigation of Generalized Shallow Water Equation for Shock Wave and Abundant Solutions Using a Further Extended Tanh Method

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Abstract: Further extended tanh (FET) method is suggested in this communication for solving generalized shallow water equation (GSWE). Based on the symbolic computational software, a realistic nonlinear integrable equation, GSWE that arises, typically, in atmospheric and ocean modelling, was reinvestigated to see its geometric feature using the FET method. The shock wave, soliton-like, kink type and periodic-like solution were found to be obtained. It is observed that the energy concentration of different wave profile is depended on the coefficient to Riccati equation. The obtained results were found to be somewhat similar with some of that obtained in the previous studies.

Keywords: *Nonlinear evolution equation, Further extended tanh method, Generalized shallow water equation*

1. Introduction

Nonlinear evolution equation (NLEE) is addressed as a very important tool to explain nonlinear phenomenon over the world [1]. It is believed the nonlinear phenomenon to have some important effects on various fields of science, especially, in fluid mechanics, plasma physics, chemical physics, optical fiber, chemical kinematics, solid-state physics, and geophysics [1]. Nonlinear partial differential equation (PDE) involves in both space and time (evolve with time), normally, known as NLEE. Recently, searching of exact solutions for NLEEs have been a major concern for the researcher, interested in the field of nonlinear physics [2]. By the virtue of the development of symbolic computational software such as Mathematica, Matlab (Mupad), Maple, researchers are being interested in finding a fully explicit solution of NLEEs using differentanalytic tools. For solving NLEEs, different methods are developed in various literatures like inverse scattering method [3], Hirota bilinear method [4], Painleve expansion [5], sine-cosine method [6], homogeneous balance method [7], homotopy perturbation method [8], [9], [10], Adomian decomposition method [11], tanh function method [12], [13], [14], further extended tanh method [2], F-expansion method [15], [16], [17], [18], [19], exp-function method [20], [21], auxiliary equation method [22], (G′/G)-expansion method [23], [24], [25], [26], [27], [28]. Among the aforementioned methods, tanh method is one of the simplest and most straightforward methods in investigating the solutions of NLEEs [29], [30]. It is claimed in their study [31] that the method can still be applied in searching solitary and shock wave profile solutions of NLEEs. As a class of NLLEs, (2+1)-dimensional (hereafter (2+1)-D) GSWE has been chosen to explore its different type solutions. The main intention of this study is to solve (2+1)-D GSWE using the FET method and to see its different geometric features for various type solution profiles such as shock wave, soliton-like, kink type and periodic solutions. The GSWE has been solved in several studies such as [1], [32], [33], [34]. But, as far the authors knowledge goes on, the equation still not studied sufficiently to see its geometric feature for different wave profiles. Furthermore, Alquran et al. [35] investigated shock-wave profile and periodic wave profile of the Vakhnenko-Parkes (VK) equation, the generalized equal width-Burgers (GEWB) equation, and the generalized regularized-long-wave (GRLW) equation using Unified and Bernoulli sub-equation method, and Tariqa and Seadawy [36] studied the (2+1)-D and (3+1)-D Kadomtsev-Petviashvili-Benjamin-Bona-Mahony (KP-BBM) equation for solitary wave, shock wave and periodic wave profiles using solitary wave ansatz method. All the mentioned equation represent water wave model, are governing equations for fluid flows also they can explain longwave phenomena. GSWE being a class of water wave model equation is studied insufficiently for shock wave solution. Whatever, FET method is the last development in the class of tanh method [2]. It is pointed out here that the beauty of the FET method is that it can explore shock wave, soliton-like and periodic solution simultaneously, in a very simple way [2]. The reason of the choice of this equation is that the equation is reported as one of the most important equations in explaining nonlinear phenomena in atmosphere and ocean dynamic modelling [1].

The rest of the part of this paper is organized as follows. Section 2 deals with the description of the further extended tanh method. The solution GSWE is presented in Section 3. Finally, the discussion of obtained results and concluding remarks are presented in section 4.

2. The further extended tanh method

For a NLEE in the independent variables $x = (t, x_1, x_2, \ldots, x_n)$ and field variable $u(x)$, the solution can be written in the following form as in [2]:

$$u = \sum_{i=0}^{n} a_i(x)\varphi^i(\omega(x)), \qquad (1)$$

With the help of Riccati equation $\varphi' = \delta(1 + \mu\varphi^2)$, (2)

where δ represents a nonzero constant coefficient and $\mu = \pm 1$, prime stands for denoting derivative with respect to ω and ω composite variable will see later. The state variable u can be explicitly determined through executing the following steps:

Step 1: Equating the highest order nonlinear terms and the highest order linear partial derivative in the given NLEE yields the value of n.

Step 2: Using Eqs. (1)-(2) in NLEE with the obtained value of n, a polynomial in $\varphi$is obtained. Equating the coefficients of the polynomial to zero, a system of PDEs in $a_i (i = 0, 1, \ldots, n)$ and ω is obtained.

Step 3: Solving this system of PDEs obtained in step 2, values of $a_i (i = 0, 1, \ldots, n)$ and ω are obtained.

Step 4: Eq. (2) processes the solution in following form

$$\varphi = \begin{cases} tanh(\delta\omega), & \mu = -1. \\ coth(\delta\omega), & \mu = -1. \\ tan(\delta\omega), & \mu = 1. \\ -cot(\delta\omega), & \mu = 1. \end{cases} \qquad (3)$$

The solution together with the values of $a_i (i = 0, 1, \ldots, n)$ and ω, we obtained different solution of the NLEEs.

3. Application of the method and obtained result

As in [34], (2+1)-D GSWE can be written in the form

$$u_{yt} + u_{xxxy} - 3u_{xx}u_y - 3u_x u_{xy} = 0. \qquad (4)$$

Balancing between the highest order derivative u_{xxxy} and nonlinear term $3u_{xx}u_y$ or $3u_x u_{xy}$ in Eq. (4) leads to $n = 1$. Thus, setting $a_0 = a(y, t)$ and $a_1 = b(y, t)$, Eq. (1) leads to following form

$$u = a(y, t) + b(y, t)\varphi(\omega), b \neq 0, \qquad (5)$$

where, $\omega = \alpha x + q(y, t)$ with α is a nonzero constant and $q(y, t)$ represents an unknown function to be determined.

Substituting Eq. (5) in coordination with Eq. (2) into Eq. (4) and equating the coefficients of polynomials in φ with zero by the help of symbolic computation software Maple 17, following system of over-determined PDEs in a, b, q, α and δ were obtained,

$$\varphi^0: a_{yt} + b_y\delta q_t + b_t\delta q_y + 2b_y\delta^3\mu\alpha^3 + b_y\delta q_{yt} - 3b\delta^2\alpha^2 b_y = 0$$

$$\varphi^1: b_{yt} - 6b\delta^2\mu\alpha^2(b\delta q_y + a_y) + 16b\delta^4\mu^2\alpha^3 q_y + 2b\delta^2\mu q_t q_y - 6b^2\delta^3\mu q_y = 0$$

$$\varphi^2: \delta\mu(b_y q_t + b_t q_y + 8b_y\delta^2\mu\alpha^3 + bq_{yt} - 12b\delta\alpha^2 b_y) = 0$$

$$\varphi^3: 2b\delta^2\mu^2(-9b\delta\alpha^2 q_y + 20\delta^2\mu\alpha^3 q_y + q_t q_y - 3\alpha^2(b\delta q_y + a_y)) = 0$$

$$\varphi^4: 3b_y\delta^2\mu^2\alpha^2(2\delta\mu\alpha - 3b) = 0$$

$$\varphi^5: 12b\delta^3\mu^3\alpha^2 q_y(2\delta\mu\alpha - b) = 0$$

To get the nontrivial solution, $b \neq 0$. Then there arise two subcases to consider.

Case 1.$b = 2\delta\mu\alpha$, $q_{yt} = 0$ and $a_{yt} = 0$ yield the following sets of solutions

$$Set1. \{a = A(t), b = 2\mu\alpha\delta, q = 4\delta^2\mu\alpha^3 t + B(y)\}.$$

Thus, the corresponding solution could be obtained in following form,

$$u(x, y, t) = \begin{cases} A(t) - 2\alpha\delta\tanh[R], & \mu = -1, \\ A(t) - 2\alpha\delta\coth[R], & \mu = -1, \\ A(t) + 2\alpha\delta\tan[R], & \mu = 1, \\ A(t) - 2\alpha\delta\cot[R], & \mu = 1, \end{cases} \qquad (6)$$

where$S = \delta\alpha x - 4\delta^3\alpha^3 t + \delta B(y)$; A is function of single variable t and B is that of single y.

$$Set2. \{a = G(t) + \frac{C_1 + 4\mu\alpha^3\delta^2}{3\alpha^2}F(y), b = 2\mu\alpha\delta, = C_1 t + F(y) + C_2\}$$

$$u(x, y, t) = \begin{cases} G(t) + SF(y) - 2\alpha\delta\tanh[T], \mu = -1, \\ G(t) + SF(y) - 2\alpha\delta\coth[T], \mu = -1, \\ G(t) + SF(y) - 2\alpha\delta\tan[T], \mu = 1, \\ G(t) + SF(y) - 2\alpha\delta\cot[T], \mu = 1 \end{cases} \qquad (7)$$

where $S = \frac{C_1 + 4\alpha^3\delta^2}{3\alpha^2}$; $R = \delta\alpha x + C_1\delta t + \delta F(y) + \delta C_2$; $F(y)$ is function of single variable y, G is that of single t, and C_1, C_2 represent arbitrary constants.

Case 2.$b = \frac{2}{3}\delta\mu\alpha$, $q_{yt} = 0$ and $a_{yt} = 0$ yield the following set of solution

$$Set3. \{a = f(t), b = \frac{2}{3}\mu\alpha\delta, q = g(t)\}$$

$$u(x, y, t) = \begin{cases} f(t) - \frac{2}{3}\alpha\delta\tanh[\delta\alpha x + \delta g(t)], \mu = -1, \\ f(t) - \frac{2}{3}\alpha\delta\coth[\delta\alpha x + \delta g(t)], \mu = -1, \\ f(t) + \frac{2}{3}\alpha\delta\tan[\delta\alpha x + \delta g(t)], \mu = 1, \\ f(t) - \frac{2}{3}\alpha\delta\cot[\delta\alpha x + \delta g(t)], \mu = 1, \end{cases} \qquad (8)$$

where f is a function of single variable t, g is that of single$y$

Thus, abundant solution including shock wave profile, soliton-like and periodic-like solutions of GSWE were obtained. For a better understanding, the shock wave profiles along with its contour are presented graphically in Figs. 1-2 generated from each of the first solution of Eqs. (6) and (7) with the proper setting the values of arbitrary functions and parameters. The periodic, soliton-like and periodic-like solutions are presented graphically in Fig. 3-5 respectively, what are generated from the fourth solution of Eq. (7), the second and the fourth solution of the Eq. (8). Forother

(a) (b)

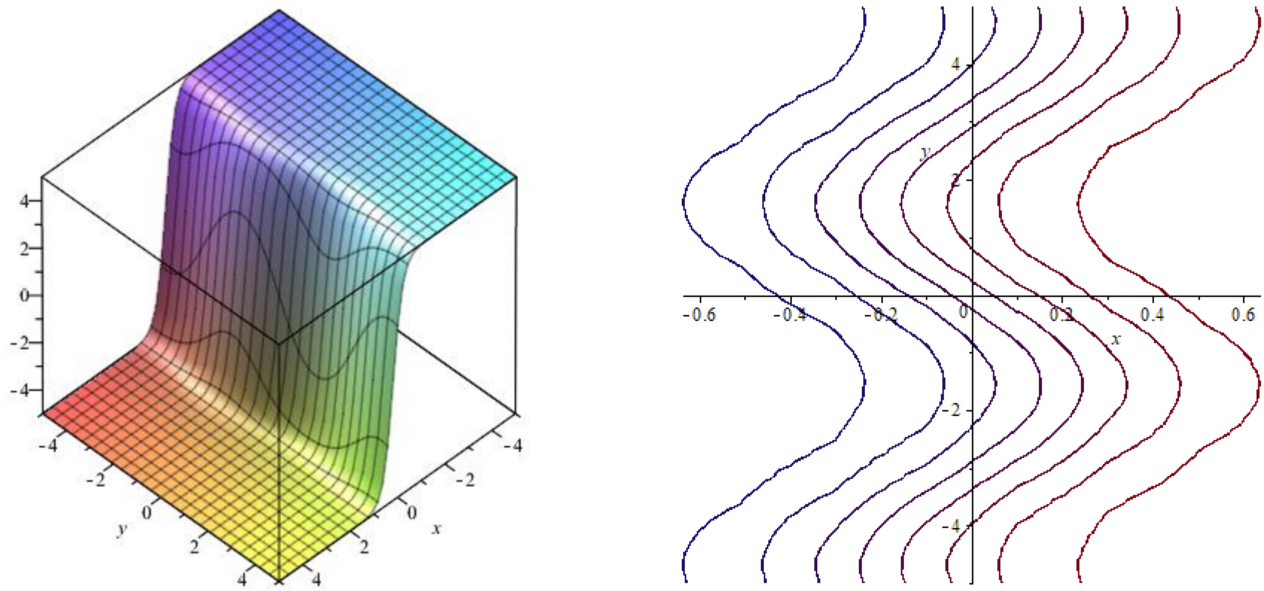

Fig. 1. The 3-dimensional (3D) representation of the shock wave profile and its contour [Obtained from the first solution of Eq. (6)] for $\alpha = 5, \delta = 0.5, A(t) = 0$ and $B(y) = sin(y)$.

(a) (b)

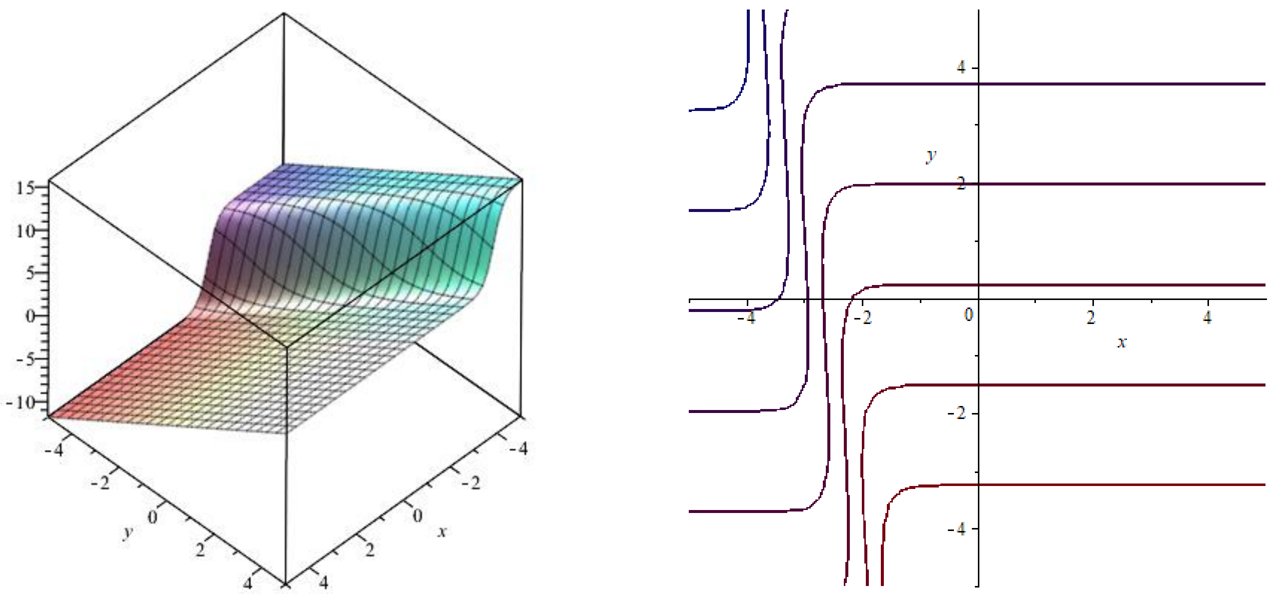

Fig. 2. The 3D representation of the shock wave profile and its contour [Obtained from the first solution of Eq. (7)] for $C_1 = 7, C_2 = 0, \alpha = 5, \delta = 0.5, F(y) = y$ and $G(t) = 2$.

other remaining solutions, the similar results are found to be obtained. So, due to the shake of brevity, the other type of wave profiles has not presented graphically.

(a) (b)

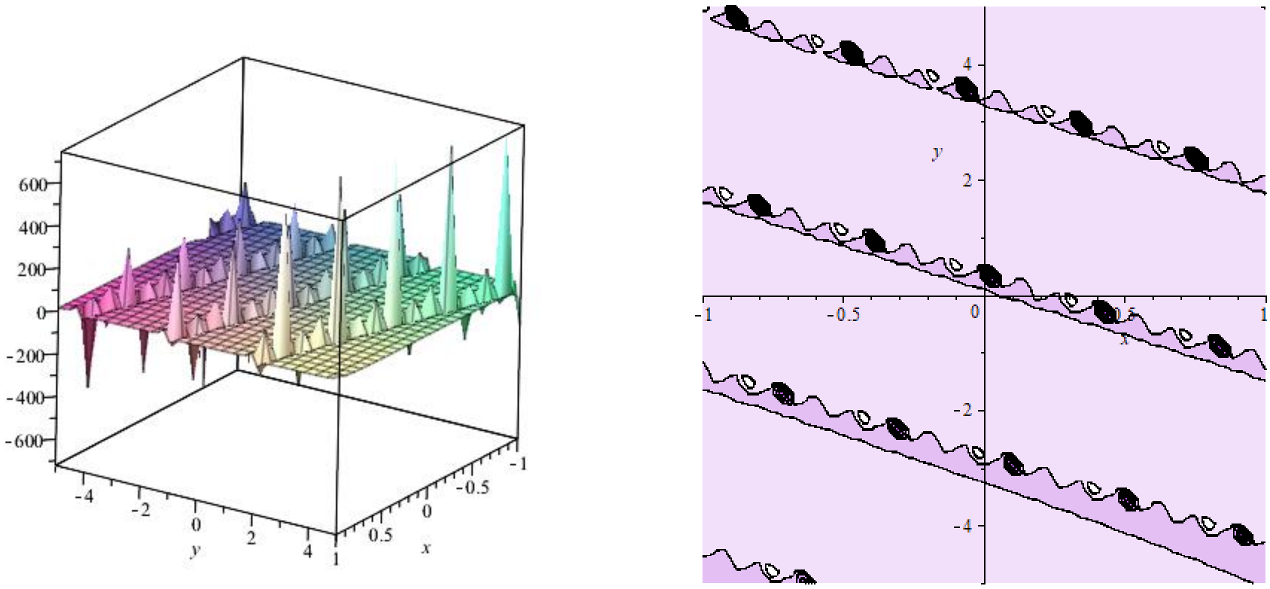

Fig. 3 The 3D representation of the periodic wave profile and its contour [Obtained from the fourth solution of Eq. (7)] at $t = 2$ for $C_1 = 7/5, C_2 = 0, \alpha = 3/2, \delta = 1, F(y) = y$ and $G(t) = t$.

(a) (b)

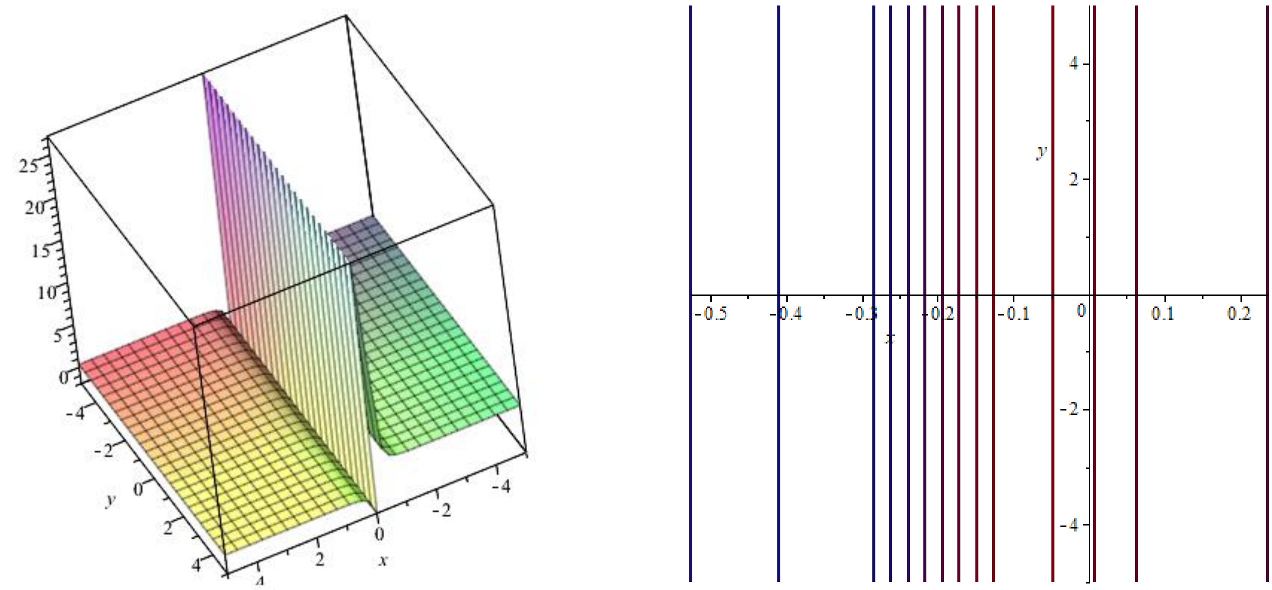

Fig. 4. The 3D soliton-like profile and its contour [Obtained from the second solution of Eq. (8)] for $\alpha = 5, \delta = 0.5, F(t) = sin(t)$ and $G(t) = 2$.

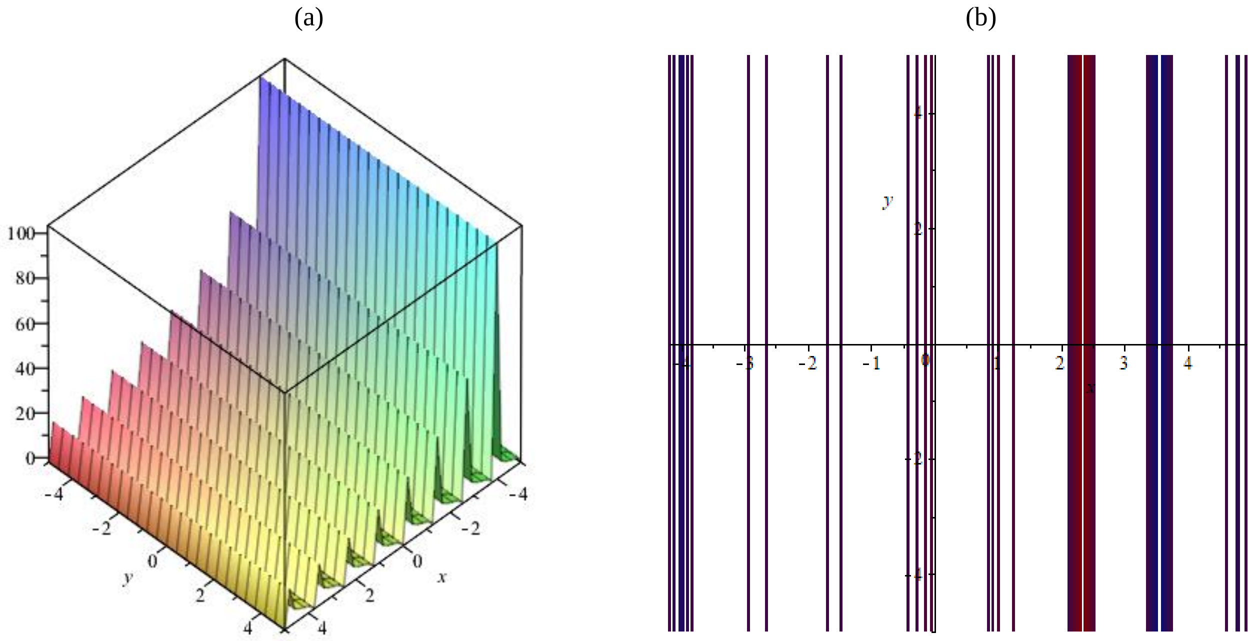

Fig. 5. The 3D periodic-like profile and its contour [Obtained from the fourth solution of Eq. (8)] for $\alpha = 5$, $\delta = 0.5$, $F(t) = \sin(t)$and $G(t) = 2$.

4. Discussion and concluding remarks

For each set, the first solution shows the shock wave profile, the second solution represents the solitary wave solution whereas the third and fourth solutions show the periodic wave profile. Sometimes periodic wave profiles are found to show kinky-periodic wave behaviors. The solutions obtained Eq. (6) are highly dependent on the parameters A and B. The use of constant values of A and Bin the first solution of Eq. (6) show always shock wave profile. Every solution obtained in Eq. (6) found to be independent of sign of δ. Setting functions instead of constant values of A and B, highly nonlinear profiles can be obtained. Due to shake of brevity, we have graphically presented only the shock wave profile from the set 1 (Eq. 6) of the solutions in Fig. 1. Equation (7) also retain four solutions including the same wave profiles mentioned above. Among them, two of them illustrated in Figs. 2 and 3. The figure (Fig. 20 shows the similar shock wave profile presented in Fig. 1 with a small change in amplitude and direction of wave propagation. Thefigure (Fig. 3) shows the kinky-periodic solitary wave solution. The both profiles are highly dependent on the value and sign of C_1 and independent of sign of δ but energy concentration of the wave profile is dependent on δ. It is to be noted here that one of the wave profiles obtained in Eq. (7) found to be similar that obtained in the study due to [34]. Similarly, soliton and periodic like solution are presented from solution symbolized by Eq. (8) in Figs. 4 and 5. The obtained solutions are found to be dependent on the sign of δ. It is worth mentioning here that our computation process and obtained profiles are about similar that one adopted in [35], [36]. The study will let us know the nonlinear behaviors in propagation of GSWE. Thus, the model can help us to understand the possible nonlinear wave profiles arisen in dynamics of GSWE and the algorithm presented in the study covers other tanh methods with no loss of straightforwardness. Thus, the algorithm could be an alternative for studying nonlinear phenomenon raising in ocean modelling and the computed wave profiles will enrich the feature of nonlinear wave profile for GSWE. The authors are interested to see the nonlinear wave pattern and their energy propagation in a fixed geometric boundary with suitable and real boundary conditions in their future work. The authors also believe that implementation of GSWE in a real geometry (such as Bay of Bengal) with proper source and boundary can explain long wave phenomena such as storm surge, tsunami waves.

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6. References

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